

COARSENING POLYHEDRAL COMPLEXES

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ABSTRACT. Given a polyhedral complex \mathcal{C} with convex support, we characterize, by a local codimension-2 condition, polyhedral complexes that coarsen \mathcal{C} . The proof of the characterization draws upon a surprising general shortcut for showing that a collection of polyhedra is a polyhedral complex and upon a property of hyperplane arrangements which is equivalent, for Coxeter arrangements, to Tits' solution to the Word Problem. The motivating special case, the case where \mathcal{C} is a complete fan, generalizes a result of Morton, Pachter, Shiu, Sturmfels, and Wienand that equates convex rank tests with semigraphoids. We also prove oriented matroid versions of our results, obtaining, as a byproduct, an oriented matroid version of Tietze's convexity theorem.

1. SUMMARY OF RESULTS

The purpose of this paper is to characterize the polyhedral complexes \mathcal{C}' that coarsen a given polyhedral complex \mathcal{C} . A **polyhedron** is a non-empty intersection of finitely many closed halfspaces. A **polyhedral complex** is a finite, nonempty collection \mathcal{C} of polyhedra such that (1) if $F \in \mathcal{C}$ and G is a face of F , then $G \in \mathcal{C}$, and (2) if F and G are in \mathcal{C} , then $F \cap G$ is a face of F and a face of G . The polyhedra in \mathcal{C} are called the **faces** of \mathcal{C} . A **fan** is (the set of nonempty faces of) a polyhedral complex all of whose nonempty faces contain the origin. Details on polyhedra, polyhedral complexes and fans can be found, for example, in [10]. We typically shorten “polyhedral complex” to “complex.” The **support** $\text{Supp}(\mathcal{C})$ of a collection \mathcal{C} of polyhedra is the union of the polyhedra in the collection. A complex has **convex support** if its support is a convex set. A complex is **complete** if its support is the entire ambient space. A complex \mathcal{C}' **coarsens** a complex \mathcal{C} if \mathcal{C}' and \mathcal{C} have the same support and if each face of \mathcal{C}' is a union of faces of \mathcal{C} .

Let \mathcal{C} be a polyhedral complex in \mathbb{R}^n with convex support. We assume that $\text{Supp}(\mathcal{C})$ is full-dimensional. (Otherwise, we restrict to the smallest affine space containing $\text{Supp}(\mathcal{C})$.) The **adjacency graph** \mathcal{G} of \mathcal{C} is the graph whose vertices are the full-dimensional faces of \mathcal{C} and whose edges are the pairs of adjacent full-dimensional faces (pairs of full-dimensional faces whose intersection is a codimension-1 face). Given a complex \mathcal{C}' coarsening \mathcal{C} , define the **edge set of \mathcal{C}'** to be the set of edges $M-N$ in \mathcal{G} such that M and N are contained in the same face of \mathcal{C}' . A complex \mathcal{C}' coarsening \mathcal{C} is uniquely determined by its edge set and vice versa. (See Section 4.) Thus to characterize complexes coarsening \mathcal{C} , we give a necessary and sufficient local condition for a set \mathcal{E} of edges of \mathcal{G} to be the edge set of a complex that coarsens \mathcal{C} .

A **polygon** in \mathcal{G} is a cycle \mathcal{P} in \mathcal{G} consisting of all of the full-dimensional faces of \mathcal{C} containing some codimension-2 face F of \mathcal{C} . If F is contained in the boundary

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of $\text{Supp}(\mathcal{C})$ then it does not define a polygon, because the set of full-dimensional faces of \mathcal{C} containing F is not a cycle. Given a codimension-2 face F of \mathcal{C} , not contained in the boundary of the convex set $\text{Supp}(\mathcal{C})$, let \mathcal{P} be the polygon in \mathcal{G} associated to F . Let $\text{Aff}(F)$ denote the **affine hull** of F , the intersection of all affine hyperplanes (hyperplanes not necessarily containing the origin) containing F . This is an affine subspace of codimension 2. Let $\text{Perp}(F)$ be the unique linear subspace orthogonal to $\text{Aff}(F)$. This 2-dimensional plane is the orthogonal complement of the linear subspace $\text{Aff}(F) - p$, where p is any point in $\text{Aff}(F)$. Choose a point x in the relative interior of F . For each face M in \mathcal{P} , define a cone $\{v \in \text{Perp}(F) : \exists \epsilon > 0 \text{ with } x + \epsilon v \in M\}$. The cones that arise in this way are the maximal cones of a complete fan $\mathcal{C}|_F$ in the plane $\text{Perp}(F)$. The adjacency graph of $\mathcal{C}|_F$ is \mathcal{P} . A set \mathcal{E} of edges in \mathcal{G} has the **polygon property** if, for every codimension-2 face F not contained in the boundary of $\text{Supp}(\mathcal{C})$, the restriction of \mathcal{E} to \mathcal{P} is the edge set of a fan in $\text{Perp}(F)$ coarsening $\mathcal{C}|_F$. We will prove the following theorem.

Theorem 1.1. *Let \mathcal{C} be a polyhedral complex with convex support and let \mathcal{G} be the adjacency graph on maximal faces of \mathcal{C} . Then a subset \mathcal{E} of the edges of \mathcal{G} is the edge set of a complex coarsening \mathcal{C} if and only if \mathcal{E} has the polygon property.*

One key ingredient in the proof of Theorem 1.1 is a useful and surprising shortcut for proving that a collection of polyhedra is a polyhedral complex. Given a collection \mathcal{M} of polyhedra, for each integer $k \geq -1$, let $\bigcap_k(\mathcal{M})$ be the union of all intersections $M \cap N$ such that $M, N \in \mathcal{M}$ and $\dim(M \cap N) \leq k$. (By convention, the empty set has dimension -1 .) For any $x \in \mathbb{R}^n$ and $\delta > 0$, let $\mathcal{B}_\delta(x)$ be the open ball $\{y \in \mathbb{R}^n : |x - y| < \delta\}$. In Section 2, we prove the following theorem, which is much more general than what is needed for Theorem 1.1.

Theorem 1.2. *Fix $k \geq -1$ and let \mathcal{M} be a finite collection of polyhedra in \mathbb{R}^n , each of dimension greater than k . Suppose:*

- (i) *For all $x \in \text{Supp}(\mathcal{M})$, there exists $\epsilon > 0$ such that, for all δ with $\epsilon > \delta > 0$, the set $[\text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M})] \cap \mathcal{B}_\delta(x)$ is path connected; and*
- (ii) *$M \cap N$ is a face of M and of N for all $M, N \in \mathcal{M}$ with $\dim(M \cap N) > k$.*

Then the collection, \mathcal{C} , of all polyhedra in \mathcal{M} and their faces is a polyhedral complex.

It is easy to see that hypothesis (i) of Theorem 1.2 holds, with $k = d - 2$, when all of the polyhedra in \mathcal{M} are d -dimensional and $\text{Supp}(\mathcal{M})$ is convex. Thus we have the following corollary, which will be used to prove Theorem 1.1.

Corollary 1.3. *Let \mathcal{M} be a finite collection of d -dimensional polyhedra. Suppose:*

- (i) *$\text{Supp}(\mathcal{M})$ is a convex set; and*
- (ii) *$M \cap N$ is a face of M and of N for all $M, N \in \mathcal{M}$ with $\dim(M \cap N) \geq d - 1$.*

Then the collection, \mathcal{C} , of all polyhedra in \mathcal{M} and their faces is a polyhedral complex.

Let $\mathcal{C}(\mathcal{A})$ be the complete polyhedral complex determined by an affine hyperplane arrangement \mathcal{A} . In Section 4, we prove Theorem 1.1, beginning with the special case where \mathcal{C} is a full-dimensional subcomplex of $\mathcal{C}(\mathcal{A})$ with convex support. The proof relies on the observation, discussed in Section 3, that every hyperplane arrangement has a property that we call path convexity. When \mathcal{A} is a Coxeter arrangement, the statement that \mathcal{A} is path convex is exactly the statement of Tits' solution [8,

Théorème 3] to the Word Problem for the corresponding Coxeter group. Essentially equivalent observations have been made in various other settings [2, 3, 6].

The broad outline of the proof of the special case of Theorem 1.1 is shared by proofs of the less general results [5, Proposition 5.2] and [4, Theorem 9]. The former, for a broad class of central hyperplane arrangements \mathcal{A} , uses a stronger condition than the polygon property, arising from the lattice theory of the weak order, to show that certain sets of edges of the adjacency graph $\mathcal{G}(\mathcal{A})$ are the edge sets of fans coarsening the fan $\mathcal{C}(\mathcal{A})$. The latter establishes Theorem 1.1 in the case $\mathcal{C} = \mathcal{C}(\mathcal{A})$, where \mathcal{A} is the Coxeter arrangement for the symmetric group (i.e. the braid arrangement). For this \mathcal{A} , the complex $\mathcal{C}(\mathcal{A})$ is the normal fan of the permutohedron, and fans coarsening $\mathcal{C}(\mathcal{A})$ are interpreted in the language of nonparametric statistics as *convex rank tests*. In [4], the edge set of a fan coarsening $\mathcal{C}(\mathcal{A})$ is described as the set of walls (codimension-1 faces) of $\mathcal{C}(\mathcal{A})$ which are removed to make the coarsening. The adjacency graph $\mathcal{G}(\mathcal{A})$ is the 1-skeleton of the permutohedron, so polygons in $\mathcal{G}(\mathcal{A})$ are squares and regular hexagons. The edge sets of fans coarsening $\mathcal{C}(\mathcal{A})$ are characterized by the *square axiom* and the *hexagon axiom*. Furthermore, edge sets satisfying the square axiom and the hexagon axiom are identified with certain conditional independence structures known as *semigraphoids*.

Theorem 1.1 in particular answers a question of Morton, Pachter, Shiu, Sturmfels, and Wienand [4, Section 1] by generalizing (in combinatorial but not probabilistic terms) the notion of a semigraphoid to an arbitrary finite Coxeter group W , as we now explain. Let \mathbf{Z} be a zonotope. A set \mathcal{E} of edges of \mathbf{Z} has the *zonotopal polygon property* if, for every $2k$ -gonal face P of \mathbf{Z} , if \mathcal{E} contains any $k - 1$ consecutive edges of P , then \mathcal{E} contains the opposite $k - 1$ consecutive edges of P . When \mathcal{A} is the central hyperplane arrangement dual to \mathbf{Z} and \mathcal{C} is the complex $\mathcal{C}(\mathcal{A})$, the 2-dimensional fans \mathcal{C}_F are defined by an arrangement of lines through the origin in $\text{Perp}(F)$. Thus the condition on edge sets reduces to the zonotopal polygon property, and we have the following corollary to Theorem 1.1.

Corollary 1.4. *Let \mathbf{Z} be a zonotope and let \mathcal{F} be the normal fan of \mathbf{Z} . Then a set \mathcal{E} of edges of \mathbf{Z} is the edge set of a fan coarsening \mathcal{F} if and only if \mathcal{E} has the zonotopal polygon property.*

The zonotopal polygon property of the usual permutohedron is equivalent to the square and hexagon axioms. Thus a generalized semigraphoid associated to a finite Coxeter group W is a set of edges of the W -permutohedron with the zonotopal polygon property. For a non-central arrangement \mathcal{A} , there is no dual zonotope, but the polygon condition on $\mathcal{G}(\mathcal{A})$ still has a simple combinatorial restatement (Lemma 4.2) for the same reason.

The proof of the special case of Theorem 1.1 also implies a local condition (Theorem 4.5) for an interior-connected union of polyhedra to be convex, which is a special case of Tietze's convexity theorem [7]. (See [9, Part IV.C].) In Section 5, we extend all of our results to the context of oriented matroids, proving, in particular, an oriented matroid version (Theorem 5.6) of Tietze's convexity theorem.

2. POLYHEDRAL COMPLEXES

In this section, we prove Theorem 1.2. We begin by establishing a well-known, easier result. (See, for example, [4, Lemma 14] or [5, Lemma 3.2].)

Lemma 2.1. *Let \mathcal{M} be a finite collection of polyhedra and let \mathcal{C} be the collection consisting of all polyhedra in \mathcal{M} and their faces. Suppose $M \cap N$ is a face of M and of N for all $M, N \in \mathcal{M}$. Then \mathcal{C} is a polyhedral complex.*

Proof. Let F, G, M , and N be polyhedra in \mathcal{C} such that F is a face of M and G is a face of N . We claim that $F \cap G$ is a face of $M \cap N$. If $F = M$ and $G = N$, then the assertion is trivial, so without loss of generality, F is a proper face of M . Let H be a hyperplane such that $H \cap M$ is the face F of M . Then also $H \cap (M \cap N) = F \cap N$ is a face of $M \cap N$. If $G = N$, then $F \cap G$ is a face of $M \cap N$, and if not, we argue similarly that $M \cap G$ is a face of $M \cap N$. Thus $F \cap G = (F \cap N) \cap (M \cap G)$ is a face of $M \cap N$, and we have proven the claim in either case. Now, since F and $F \cap G$ are faces of $M \cap N$, $F \cap G$ is a face of F . Symmetrically, $F \cap G$ is a face of G . \square

Proof of Theorem 1.2. We will verify the hypotheses of Lemma 2.1. Let M and N be distinct polyhedra in \mathcal{M} , let F be the polyhedron $M \cap N$, and let d be the dimension of F . Then there exists a point $x \in F$ such that x is not contained in any k -dimensional polyhedron in \mathcal{C} . By hypothesis (i), there exists $\epsilon > 0$ such, for all δ with $\epsilon > \delta > 0$, the set $(\text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M})) \cap \mathcal{B}_\delta(x)$ is path connected. Every face of \mathcal{C} not containing x is some positive distance from x and there are finitely many faces of \mathcal{C} . Thus there exists δ with $\epsilon > \delta > 0$ such that every face of \mathcal{C} intersecting $\mathcal{B}_\delta(x)$ actually contains x .

Now x is in M and in N , so $\mathcal{B}_\delta(x)$ intersects the relative interiors of M and N . Let $y \in (\mathcal{B}_\delta(x) \cap \text{relint}(M))$ and $z \in (\mathcal{B}_\delta(x) \cap \text{relint}(N))$. Let $\alpha : [0, 1] \rightarrow (\text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M})) \cap \mathcal{B}_\delta(x)$ be a path from y to z . We will use α to construct a sequence $M = M_0, \dots, M_j = N$ of polyhedra in \mathcal{M} such that, for each $i = 1, \dots, j$, the intersection $M_{i-1} \cap M_i$ is of dimension greater than k . The set M is closed, so $\alpha^{-1}(M)$ is a closed subset of $[0, 1]$. If t_1 is the maximum of the set $\alpha^{-1}(M)$, then $\alpha(t_1)$ is in M and in some $M_1 \in \mathcal{M}$. But $\alpha(t_1) \in \text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M})$, so $M \cap M_1$ has dimension greater than k . If $M_1 \neq N$, then repeat the construction to find $M_2 \in \mathcal{M}$ such that $\dim(M_1 \cap M_2) > k$, and continue until $M_j = N$.

We now show that for any sequence M_0, \dots, M_j of polyhedra in \mathcal{M} such that $M_{i-1} \cap M_i$ is of dimension greater than k for each $i = 1, \dots, j$, the intersection $M_0 \cap \dots \cap M_j$ is a face of M_j . We argue by induction on j , the case $j = 0$ being trivial. If $j > 0$, then by induction $M_0 \cap \dots \cap M_{j-1}$ is a face G of M_{j-1} . Now $G' = M_{j-1} \cap M_j$ is a face of M_{j-1} and of M_j by hypothesis (ii). Thus $M_0 \cap \dots \cap M_j$ is a face of M_{j-1} because it is the intersection of two faces, G and G' , of M_{j-1} . But then $M_0 \cap \dots \cap M_j$ is a face of G' , and thus a face of M_j .

We have shown that $F' = M_0 \cap \dots \cap M_j$ is a face of N . Since α is contained in $\mathcal{B}_\delta(x)$, each M_i intersects $\mathcal{B}_\delta(x)$, so by the definition of δ , each M_i contains x . Thus F' contains x , so by the definition of x , F' is face of \mathcal{C} of dimension at least d . However, F' is contained in the d -dimensional polyhedron $F = M \cap N$, so F' has dimension d . Let H be a hyperplane such that $N \cap H$ is F' . Since $F' \subseteq F$ and both are d -dimensional polyhedra, we have $F \subseteq H$. Thus $F \subseteq (N \cap H) = F'$. Therefore $F' = F$, so that F is a face of N . By symmetry, F is a face of M . \square

3. PATH CONVEXITY

In this section, we show that every hyperplane arrangement has a property that we call path convexity. This fact will be crucial in the proof of Theorem 1.1.

A **hyperplane arrangement** in \mathbb{R}^n is a finite collection \mathcal{A} of affine hyperplanes. The closures of the connected components of $\mathbb{R}^n \setminus (\bigcup_{H \in \mathcal{A}} H)$ are called **regions**. The regions are the maximal faces of a complete polyhedral complex $\mathcal{C}(\mathcal{A})$. Let $\mathcal{G}(\mathcal{A})$ be the adjacency graph of the complex $\mathcal{C}(\mathcal{A})$.

Let $Q, R \in \mathcal{R}(\mathcal{A})$. A **path** in $\mathcal{G}(\mathcal{A})$ from Q to R is a sequence R_0, R_1, \dots, R_k of regions with $Q = R_0$ and $R = R_k$, such that $R_{i-1} - R_i$ is an edge in $\mathcal{G}(\mathcal{A})$ for each i from 1 to k . The length of a path R_0, R_1, \dots, R_k is k , one less than the number of entries in the sequence. A **braid move** on a path alters the path by deleting an adjacent subsequence Q_0, \dots, Q_m from the path and replacing it with a sequence Q'_0, \dots, Q'_m such that $Q_0 = Q'_0$, $Q_m = Q'_m$ and the cycle $Q_0, Q_1, \dots, Q_m, Q'_{m-1}, \dots, Q'_0$ is a polygon in $\mathcal{G}(\mathcal{A})$. A braid move does not change the length of the path. A **nil move** on paths alters a path by replacing an adjacent subsequence Q_0, Q_1, Q_2 such that $Q_0 = Q_2$ by the singleton sequence Q_0 .

We say that a path γ is **reduced** if it has minimal length among all paths from Q to R . The arrangement \mathcal{A} is **path convex** if, for every pair Q, R of regions in \mathcal{A} , every path γ from Q to R , and every reduced path ρ from Q to R , the path γ can be transformed, by a sequence of braid moves and nil moves, to the path ρ . The appropriateness of the term “convex” in this definition will become apparent in the proof of Theorem 1.1, particularly in Lemma 4.3.

Theorem 3.1. *Every hyperplane arrangement is path convex.*

A slightly weaker statement for oriented matroids is [1, Proposition 4.4.6]. We now prepare to prove Theorem 3.1. Given $Q, R \in \mathcal{R}(\mathcal{A})$, let $S(Q, R)$ be the set of hyperplanes of \mathcal{A} that separate Q from R . The following lemma is well known.

Lemma 3.2. *A path from Q to R is reduced if and only if its length is $|S(Q, R)|$.*

Proof. Moving from one region to an adjacent region, one crosses exactly one hyperplane of \mathcal{A} . Thus a path from Q to R has length at least $|S(Q, R)|$. If x is a generic point in the interior of Q and y is a generic point in the interior of R , then the line segment \overline{xy} intersects each hyperplane in $S(Q, R)$ exactly once, intersects no two hyperplanes in $S(Q, R)$ in the same point, and intersects no hyperplane of $\mathcal{A} \setminus S(Q, R)$. Thus \overline{xy} defines a path of length $|S(Q, R)|$ from Q to R . \square

The arrangement \mathcal{A} is **reduced-path connected** if, for every pair Q, R of regions in \mathcal{A} and every pair γ, ρ of reduced paths from Q to R , the path γ can be transformed, by a sequence of braid moves, to the path ρ .

Lemma 3.3. *If \mathcal{A} is reduced-path connected, then \mathcal{A} is path convex.*

Proof. Suppose \mathcal{A} is reduced-path connected. Let $\gamma = (R_0, R_1, \dots, R_m)$ be any path from Q to R and let ρ be any reduced path from Q to R .

If γ is not reduced, then Lemma 3.2 says that $m > |S(P, Q)|$. Thus there exists a smallest positive integer k such that $k > |S(R_0, R_k)|$. Then R_0, R_1, \dots, R_{k-1} is a reduced path and $|S(R_0, R_k)| = k - 2$. By Lemma 3.2, there is a reduced path $R'_0, R'_1, \dots, R'_{k-2}$ from R_0 to R_k , and thus the path $R'_0, R'_1, \dots, R'_{k-2}, R_{k-1}$ is reduced. (Notice that an **unprimed** R_{k-1} is the last region in this path.) Since \mathcal{A} is reduced-path connected, there is a sequence of braid moves that transforms R_0, R_1, \dots, R_{k-1} to $R'_0, R'_1, \dots, R'_{k-2}, R_{k-1}$. The same braid moves transform the path γ to $R'_0, R'_1, \dots, R'_{k-2}, R_{k-1}, R_k, \dots, R_m$. But $R'_{k-2} = R_k$, so a nil move can be applied to $R'_0, R'_1, \dots, R'_{k-2}, R_{k-1}, R_k, \dots, R_m$, replacing R'_{k-2}, R_{k-1}, R_k with R_k .

Repeating the process, we transform γ to a reduced path γ' by a sequence of braid moves and nil moves. By the reduced-path connectedness of \mathcal{A} , γ' can be transformed to ρ by a sequence of braid moves. \square

Lemma 3.3 reduces Theorem 3.1 to the following theorem.

Theorem 3.4. *Every hyperplane arrangement is reduced-path connected.*

Theorem 3.4 was proved by Deligne [3, Proposition 1.12] for simplicial hyperplane arrangements, by Salvetti [6, Lemma 11], and also by Cordovil and Moreira [2, Theorem 2.4] for oriented matroids. For the sake of completeness, we give a short proof which is similar to the argument given in [2, 6].

Proof of Theorem 3.4. Let $\gamma = (Q_0, Q_1, \dots, Q_k)$ and $\rho = (R_0, R_1, \dots, R_k)$ be reduced paths with $Q_0 = R_0$ and $Q_k = R_k$. We will show that γ and ρ are related by a sequence of braid moves.

Let F_0, F_1, \dots, F_m be a sequence of facets (maximal proper faces) of Q_0 , chosen to minimize m subject to the following requirements: (1) that $F_0 = Q_0 \cap Q_1$, (2) that $F_m = Q_0 \cap R_1$, (3) that $F_{i-1} \cap F_i$ has codimension 2 for each $i = 1, \dots, m$, and (4) that, for each $i = 0, 1, \dots, m$, the hyperplane H_i containing F_i is in the set $S(Q_0, Q_k)$. We will show that such a sequence F_0, F_1, \dots, F_m exists. For each $i = 1, \dots, k$, choose a point $x_i \in Q_{i-1} \cap Q_i$ and a point $y_i \in R_{i-1} \cap R_i$ and concatenate the segments $\overline{x_1 x_2}, \dots, \overline{x_{k-1} x_k}, \overline{x_k y_k},$ and $\overline{y_k y_{k-1}}, \dots, \overline{y_2 y_1}$ to construct a continuous curve $\alpha : [0, 1] \rightarrow \mathbb{R}^n$ that begins in $Q_0 \cap Q_1$, passes through $Q_1, \dots, Q_k, R_{k-1}, \dots, R_1$, ending in $Q_0 \cap R_1$. Choose a point x in the relative interior of Q_0 . Define a continuous curve β in the boundary of Q_0 by taking $\beta(t)$ to be the unique point on the boundary of Q_0 and on the line segment with endpoints x and $\alpha(t)$. Let U be the union of all lines that contain x and that intersect a face of Q_0 of codimension 3 or greater. Since U has codimension 2, for generic choices of the x_i and y_i , the path α avoids U . Thus β avoids faces of Q_0 of codimension 3 or greater, so β defines a sequence F_0, F_1, \dots, F_m of facets of Q_0 satisfying requirements (1), (2), and (3). To see that the sequence satisfies requirement (4), note that each H_i intersects a line segment connecting a point in the interior of some Q_j or R_j , for $1 \leq j \leq k$, to the point $x \in \text{int}(Q_0)$. Thus $H_i \in S(Q_0, Q_j)$ or $H_i \in S(Q_0, R_j)$ for some $1 \leq j \leq k$. But since γ and ρ are reduced paths, we have $S(Q_0, Q_1) \subset S(Q_0, Q_2) \subset \dots \subset S(Q_0, Q_k)$, and $S(Q_0, R_1) \subset S(Q_0, R_2) \subset \dots \subset S(Q_0, R_k)$, so $H_i \in S(Q_0, Q_k)$.

We now argue by induction on k and on m . If $k = 0$, then $Q_0 = Q_k$ and the assertion is trivial. Now suppose $k > 0$. If $m = 0$, then $F_0 = F_m$, so $Q_1 = R_1$. By induction on k , there is a sequence of braid moves relating Q_1, \dots, Q_k to R_1, \dots, R_k . This same sequence of braid moves relates γ to ρ .

Now suppose $m > 0$ as well. Let \mathcal{A}' be the set of hyperplanes in \mathcal{A} containing the codimension-2 face $F_{m-1} \cap F_m$. Any region T with $\{H_{m-1}, H_m\} \subseteq S(Q_0, T)$ has $\mathcal{A}' \subseteq S(Q_0, T)$. In particular, $\mathcal{A}' \subseteq S(Q_0, Q_k)$. Let x be a point in the relative interior of Q_0 and let p be a point in the relative interior of $F_{m-1} \cap F_m$. For small enough $\epsilon > 0$, the point $p + \epsilon(p - x)$ is in a region T with $S(Q_0, T) = \mathcal{A}'$. Let μ be a reduced path from T to Q_k . Then μ has length $|S(Q_0, Q_k)| - |\mathcal{A}'|$ because $S(T, Q_k) = S(Q_0, Q_k) \setminus \mathcal{A}'$. There are two reduced paths from Q_0 to T , related by a braid move involving the polygon dual to $F_{m-1} \cap F_m$. Concatenating these paths with μ , we obtain a reduced path ρ' from Q_0 to Q_k starting with the regions Q_0, R_1 , and a reduced path γ' from Q_0 to Q_k starting with Q_0 and then continuing

to the region which shares the facet F_{m-1} with Q_0 . By induction on m , the paths γ and γ' are related by a sequence of braid moves. By construction, γ' and ρ' are related by a single braid move. Let $\rho' = R'_0, R'_1, \dots, R'_k$, so that $R'_0 = R_0 = Q_0$, $R'_1 = R_1$, and $R'_k = R_k = Q_k$. By induction on k , R'_1, \dots, R'_k and R_1, \dots, R_k are related by a sequence of braid moves, so ρ' and ρ are related by the same sequence of braid moves. We have found a sequence of braid moves relating γ and ρ . \square

4. EDGE SETS OF COARSENING

In this section, we prove Theorem 1.1. One direction of the theorem is easy. Indeed, suppose that \mathcal{C}' is a complex coarsening \mathcal{C} such that the edge set of \mathcal{C}' fails the polygon property at the polygon defined by some codimension-2 face F of \mathcal{C} . Then it is easy to find either a maximal face M of \mathcal{C}' that is not convex, or a pair of maximal faces C and D of \mathcal{C}' , each having F in their boundary, such that $C \cap D$ is not a face of \mathcal{C} . This contradiction proves the “only if” assertion of Theorem 1.1. To prove the converse, we begin by proving the following lemma, which implies in particular that a full-dimensional complex with convex support is determined entirely by its set of full-dimensional faces. The lemma justifies the assertion, made in Section 1, that coarsenings of such a complex are determined by their edge sets.

Lemma 4.1. *If \mathcal{C} is a polyhedral complex such that $\text{Supp}(\mathcal{C})$ is full-dimensional and convex, then every maximal face of \mathcal{C} is full-dimensional.*

Proof. Let F be a face of \mathcal{C} and let x be a point in the relative interior of F . Since the support of \mathcal{C} is full-dimensional and convex, any open ball about x intersects the interior of the support of \mathcal{C} . Thus if F is not full-dimensional, any open ball about x intersects some polyhedron in $\mathcal{C} \setminus \{F\}$. Since \mathcal{C} is finite and faces of \mathcal{C} are closed, some face G in $\mathcal{C} \setminus \{F\}$ actually contains x . Since \mathcal{C} is a complex, $F \cap G$ is a face of F and of G , but since this intersection contains a point x in the relative interior of F , the intersection contains F , and we conclude that F is a proper face of G . The lemma follows. \square

Let \mathcal{A} be a hyperplane arrangement and continue the notation of Section 3. Until further notice, \mathcal{C} will be a full-dimensional subcomplex of $\mathcal{C}(\mathcal{A})$ with convex support. We first prove Theorem 1.1 for this special choice of \mathcal{C} . The following lemma is immediate. (See the paragraph before Corollary 1.4.)

Lemma 4.2. *If \mathcal{C} is a full-dimensional subcomplex of $\mathcal{C}(\mathcal{A})$, then every polygon in \mathcal{G} has evenly many edges. A set \mathcal{E} of edges of \mathcal{G} has the polygon property if the following condition holds for every $2k$ -gon \mathcal{P} in \mathcal{G} : If \mathcal{E} contains any $k - 1$ consecutive edges of \mathcal{P} , then \mathcal{E} contains the opposite $k - 1$ consecutive edges of \mathcal{P} .*

We say that \mathcal{E} has the **weak polygon property** if the following condition holds for each $2k$ -gon \mathcal{P} in \mathcal{G} : If \mathcal{E} contains any k consecutive edges of \mathcal{P} , then \mathcal{E} contains all of the edges of \mathcal{P} . The polygon property implies the weak polygon property.

A **pre-complex** \mathcal{C} is a collection of polyhedra such that if $F \in \mathcal{C}$ and G is a face of F , then $G \in \mathcal{C}$. The concept of **coarsening** for pre-complexes can be defined exactly as for complexes in Section 1. We first show that the weak polygon property characterizes pre-complexes coarsening \mathcal{C} .

Lemma 4.3. *Let \mathcal{E} be a set of edges in \mathcal{G} with the weak polygon property. Then \mathcal{E} is the edge set of a pre-complex \mathcal{C}' coarsening \mathcal{C} . The maximal faces of \mathcal{C}' are full-dimensional and have pairwise disjoint interiors.*

Proof. We will think of \mathcal{E} not only as a set of edges of \mathcal{G} , but as a graph in its own right, whose vertices are the full-dimensional faces of \mathcal{C} . Each connected component of \mathcal{E} is in particular a set of regions. Consider the union of this set of regions. We must show that each such union is a polyhedron. In fact, we need only show convexity; the fact that the union is a polyhedron will follow.

Let M be a such a union. Choose points $x, y \in M$. We will show that the line segment \overline{xy} is contained in M . If there is a region R of \mathcal{A} with $\{x, y\} \subseteq R \subseteq M$, then $\overline{xy} \subseteq R \subseteq M$, so suppose x and y are not contained in the same region. Let Q be a region with $x \in Q \subseteq M$ and let R be a region with $y \in R \subseteq M$. Suppose for the moment that $x \in \text{int}(Q)$ and $y \in \text{int}(R)$ and that the line segment \overline{xy} does not intersect any face of $\mathcal{C}(\mathcal{A})$ of codimension greater than 1. Then, as in the proof of Lemma 3.2, \overline{xy} defines a path ρ of length $|S(Q, R)|$ from Q to R . By Lemma 3.2, ρ is a reduced path. On the other hand, since Q and R are both contained in M , there is a path $\gamma = (R_0, R_1, \dots, R_k)$ from Q to R which is not only a path in $\mathcal{G}(\mathcal{A})$, but also a path in \mathcal{E} . By the path convexity of \mathcal{A} , the path γ can be converted to the path ρ by a sequence of braid moves and nil moves. Trivially, each nil move applied to γ produces a new path in \mathcal{E} . Furthermore, the weak polygon property of \mathcal{E} implies that, when a braid move is performed on γ , the new path is also a path in \mathcal{E} . We conclude that ρ is a path in \mathcal{E} . In particular, each region in ρ is contained in M , so $\overline{xy} \subseteq M$.

If $x \notin \text{int}(Q)$, if $y \notin \text{int}(R)$ and/or if \overline{xy} intersects a face of $\mathcal{C}(\mathcal{A})$ of codimension greater than 1, then there exist points $x' \in \text{int}(Q)$ and $y' \in \text{int}(R)$, arbitrarily close to x and y respectively, such that $\overline{x'y'}$ does not intersect any face of $\mathcal{C}(\mathcal{A})$ of codimension greater than 1. Therefore each point on \overline{xy} is arbitrarily close to a point which we have proven to be in M . Since M is a union of finitely many closed polyhedra, it is closed, so $\overline{xy} \subseteq M$.

The collection of all such polyhedra M and their nonempty faces is a pre-complex \mathcal{C}' coarsening $\mathcal{C}(\mathcal{A})$. By construction, the maximal faces of \mathcal{C}' are full-dimensional and have pairwise disjoint interiors. It remains to show that \mathcal{E} is indeed the edge set of \mathcal{C}' . By construction, \mathcal{E} is contained in the edge set of \mathcal{C}' . Suppose $\{Q, R\}$ is a pair of regions contained in the same maximal face of \mathcal{C}' . Then the path Q, R is a reduced path ρ (in $\mathcal{G}(\mathcal{A})$) from Q to R . Since Q and R are contained in the same maximal face of \mathcal{C}' , there exists a path γ in \mathcal{E} from Q to R . Arguing as above, the path ρ is also a path in \mathcal{E} , or in other words, $Q-R$ is an edge in \mathcal{E} . We have shown that \mathcal{E} is the edge set of \mathcal{C}' . \square

Lemma 4.4. *Let \mathcal{E} have the polygon property, so that \mathcal{E} is the edge set of a pre-complex \mathcal{C}' coarsening \mathcal{C} , by Lemma 4.3. If M and N are maximal faces of \mathcal{C}' and $M \cap N$ has codimension 1, then $M \cap N$ is a face of M and a face of N .*

Proof. Suppose M and N are maximal faces of \mathcal{C}' and $M \cap N$ has codimension 1. Since M and N are unions of faces of $\mathcal{C}(\mathcal{A})$, their intersection is a union of codimension-1 faces of $\mathcal{C}(\mathcal{A})$. Since M and N are convex and have disjoint interiors, $M \cap N$ is contained in some face F of M of codimension 1. The face F is also a union of codimension-1 faces of $\mathcal{C}(\mathcal{A})$. We now prove the following claim: If G is a codimension-1 face of $\mathcal{C}(\mathcal{A})$ contained in $M \cap N$ and G' is a codimension-1 face of $\mathcal{C}(\mathcal{A})$ contained in F such that $G \cap G'$ has codimension 2, then $G' \subseteq (M \cap N)$.

To prove the claim, let \mathcal{P} be the polygon consisting of regions containing the codimension-2 face $G \cap G'$ of $\mathcal{C}(\mathcal{A})$. Since G and G' are both in the face F of M , the polygon \mathcal{P} contains a region R with $G \subset R \subset M$ and a region R' with $G' \subset R' \subset M$.

Since G and G' are both contained in the hyperplane defining F as a codimension-1 face of M , we can name the regions of \mathcal{P} as the cycle $R_0, R_1, \dots, R_{2k} = R_0$ with $R_1 = R$ and $R_k = R'$. Since R_1 and R_k are both in M , and since M is convex, we conclude that the regions R_1, \dots, R_k are all in M . Now the polygon property and Lemma 4.2 imply that the regions R_{k+1}, \dots, R_{2k} are all in the same maximal face of \mathcal{C}' . But $R_{2k} \subset N$ because $G \subset N$. Thus in particular $R_{k+1} \subset N$, so that $G' \subset N$. By hypothesis, $G' \subset F \subset M$, so $G' \subset M \cap N$. We have established the claim.

Given any two codimension-1 faces G and G' of $\mathcal{C}(\mathcal{A})$ contained in F , choosing generic points $x \in G$ and $x' \in G'$, the line segment $\overline{xx'}$ defines a sequence $G = G_0, G_1, \dots, G_k = G'$ of codimension-1 faces of $\mathcal{C}(\mathcal{A})$, contained in F , such that G_{i-1} and G_i share a codimension-2 face for each $i = 1, \dots, k$. Since there exists a codimension-1 face of $\mathcal{C}(\mathcal{A})$ in $M \cap N$, the claim and a simple induction on k establish that $F \subseteq M \cap N$. Thus $F = M \cap N$, so that $M \cap N$ is a face of M . By symmetry, $M \cap N$ is a face of N as well. \square

Lemmas 4.3 and 4.4 and Corollary 1.3 prove Theorem 1.1 in the special case where \mathcal{C} is a full-dimensional subcomplex of $\mathcal{C}(\mathcal{A})$ with convex support. Now let \mathcal{C} be any full-dimensional polyhedral complex with convex support and let \mathcal{A} be the arrangement of hyperplanes consisting of hyperplanes containing codimension-1 faces of \mathcal{C} . Then every polyhedron in \mathcal{C} is a union of faces of $\mathcal{C}(\mathcal{A})$. Let $\bar{\mathcal{C}}$ be the subcomplex of $\mathcal{C}(\mathcal{A})$ consisting of faces contained in $\text{Supp}(\mathcal{C})$. Then \mathcal{C} coarsens $\bar{\mathcal{C}}$.

Continue the notation \mathcal{G} for the adjacency graph on maximal faces of \mathcal{C} , and let $\bar{\mathcal{G}}$ be the adjacency graph on full-dimensional faces of $\bar{\mathcal{C}}$. Let \mathcal{E} be a set of edges of \mathcal{G} with the polygon property. We now define a set $\bar{\mathcal{E}}$ of edges of $\bar{\mathcal{G}}$, describing $\bar{\mathcal{E}}$ as a set of codimension-1 faces of $\bar{\mathcal{C}}$. Each such face represents the pair of regions containing it. Let \bar{F} be a codimension-1 face of $\bar{\mathcal{C}}$, not contained in the boundary of $\text{Supp}(\mathcal{C})$. If \bar{F} is contained in a codimension-1 face F of \mathcal{C} , then choose \bar{F} to be an edge in $\bar{\mathcal{E}}$ if and only if F is an edge in \mathcal{E} . Otherwise, since \mathcal{C} coarsens $\bar{\mathcal{C}}$, \bar{F} is contained in a full-dimensional face of \mathcal{C} and intersects the interior of that full-dimensional face. In this case, choose \bar{F} to be an edge in $\bar{\mathcal{E}}$.

Let $\bar{\mathcal{P}}$ be a polygon in $\bar{\mathcal{G}}$, associated to a codimension-2 face \bar{F} of $\bar{\mathcal{C}}$. If \bar{F} is contained in a codimension-2 face F of \mathcal{C} , then $\text{Aff}(\bar{F})$ and $\text{Aff}(F)$ coincide. Since \mathcal{E} has the polygon property, it defines a fan in $\text{Perp}(F)$ which coarsens $\mathcal{C}|_F$. But $\bar{\mathcal{E}}$ defines the same fan in $\text{Perp}(\bar{F}) = \text{Perp}(F)$, so $\bar{\mathcal{E}}$ has the polygon property at $\bar{\mathcal{P}}$. If \bar{F} is contained in a codimension-1 face F of \mathcal{C} and intersects the relative interior of F , then $\bar{\mathcal{P}}$ can be written as a cycle $Q_1, \dots, Q_k, R_k, \dots, R_1, Q_1$, where Q_1 through Q_k are on one side of the hyperplane containing F , while R_1 through R_k are on the other side. Furthermore, Q_1, \dots, Q_k is a path in $\bar{\mathcal{E}}$ and R_k, \dots, R_1 is also a path in $\bar{\mathcal{E}}$. If F is an edge of \mathcal{E} , then both $Q_1 - R_1$ and $Q_k - R_k$ are edges in $\bar{\mathcal{E}}$. Otherwise, neither $Q_1 - R_1$ nor $Q_k - R_k$ is an edge in $\bar{\mathcal{E}}$. In either case, the restriction of $\bar{\mathcal{E}}$ to $\bar{\mathcal{P}}$ defines a fan coarsening $\bar{\mathcal{C}}|_{\bar{\mathcal{P}}}$. Since \mathcal{C} coarsens $\bar{\mathcal{C}}$, the only possibility remaining is that \bar{F} is contained in, and intersects the interior of, a full-dimensional face F of \mathcal{C} . In this case, the restriction of $\bar{\mathcal{E}}$ to $\bar{\mathcal{P}}$ consists of all edges of $\bar{\mathcal{P}}$. In all cases, $\bar{\mathcal{E}}$ has the polygon property at $\bar{\mathcal{P}}$. Since $\bar{\mathcal{P}}$ was chosen arbitrarily, $\bar{\mathcal{E}}$ has the polygon property.

By the special case of Theorem 1.1 already proved, $\bar{\mathcal{E}}$ is the edge set of a complex \mathcal{C}' coarsening $\bar{\mathcal{C}}$. For each maximal face M of \mathcal{C} , every pair of adjacent regions of \mathcal{A} contained in M is an edge in $\bar{\mathcal{E}}$, so \mathcal{C}' coarsens \mathcal{C} as well. We will complete the

proof by showing that \mathcal{E} is the edge set of \mathcal{C}' , as a coarsening of \mathcal{C} . Let M and N be adjacent maximal faces of \mathcal{C} . If $M-N$ is an edge in \mathcal{E} then $M \cap N$ is a union of codimension-1 faces of $\bar{\mathcal{C}}$, each of which defines an edge in $\bar{\mathcal{E}}$, so $M-N$ is in the edge set of \mathcal{C}' as a coarsening of \mathcal{C} . If not, then $M \cap N$ is a union of codimension-1 faces of $\bar{\mathcal{C}}$ none of which defines an edge in $\bar{\mathcal{E}}$, so $M-N$ is not in the edge set of \mathcal{C}' as a coarsening of \mathcal{C} . We have proved the general case of Theorem 1.1.

We conclude the section by adapting the above arguments to prove a special case of Tietze's convexity theorem. This will be generalized to oriented matroids as Theorem 5.6.

Theorem 4.5. *Let \mathcal{M} be a finite set of n -dimensional polyhedra in \mathbb{R}^n . Suppose:*

- (i) *The interior of $\text{Supp}(\mathcal{M})$ is path-connected; and*
- (ii) *For every x in the boundary of $\text{Supp}(\mathcal{M})$, there exists a closed halfspace H^+ bounded by a hyperplane H such that $x \in H$ and $\bigcup_{M \in \mathcal{M}} M \subseteq H^+$.*

Then $\text{Supp}(\mathcal{M})$ is convex.

Proof. Let \mathcal{A} be the set of hyperplanes that contain codimension-1 faces of polyhedra in \mathcal{M} . Let $\bar{\mathcal{M}}$ be the set of regions of \mathcal{A} contained in $\text{Supp}(\mathcal{M})$. Let $\bar{\mathcal{E}}$ be the set of edges $Q-R$ in $\mathcal{G}(\mathcal{A})$ such that Q and R are both in $\bar{\mathcal{M}}$. Let U be the union of all faces of codimension at least 2 of polyhedra of $\bar{\mathcal{M}}$. Since the interior of $\text{Supp}(\mathcal{M}) = \text{Supp}(\bar{\mathcal{M}})$ is path-connected and full-dimensional, for any two polyhedra M and N in $\bar{\mathcal{M}}$, there exists a continuous path in $\text{int}(\text{Supp}(\bar{\mathcal{M}})) \setminus U$ from $\text{int}(M)$ to $\text{int}(N)$. This implies that $\bar{\mathcal{M}}$ is a connected component of $\bar{\mathcal{E}}$.

Let \mathcal{P} be a $2k$ -gon in $\mathcal{G}(\mathcal{A})$ defined by a codimension-2 face F of $\mathcal{C}(\mathcal{A})$. Suppose that k consecutive edges of \mathcal{P} are in $\bar{\mathcal{E}}$ but that not all of the edges of \mathcal{P} are in $\bar{\mathcal{E}}$. Then any point x in the relative interior of F provides a violation of hypothesis (ii). Thus $\bar{\mathcal{E}}$ has the weak polygon property, so $\bar{\mathcal{E}}$ defines a pre-complex \mathcal{C}' by Lemma 4.3 (with $\mathcal{C} = \mathcal{C}(\mathcal{A})$). Since $\bar{\mathcal{M}}$ is a connected component of $\bar{\mathcal{E}}$, one of the full-dimensional polyhedra in \mathcal{C}' is $\text{Supp}(\bar{\mathcal{M}}) = \text{Supp}(\mathcal{M})$. \square

5. POLYHEDRAL COMPLEXES IN ORIENTED MATROIDS

In this section, we extend our results to the context of oriented matroids. We base our approach to oriented matroids on [1, Chapter 4].

Let $\mathcal{L} \subseteq \{+, -, 0\}^E$ be the set of covectors of an oriented matroid with no loops, over a finite ground set E . The symbol \mathcal{L} will also denote the corresponding ("big") face lattice. A **closed halfspace** in \mathcal{L} is a set H_e^+ consisting of all covectors in \mathcal{L} having component $+$ or 0 in position e , or a set H_e^- consisting of all covectors having component $-$ or 0 in position e . (In [1, Chapter 4.2], a different notion of halfspaces appears, but translating between the two conventions is easy.) A **hyperplane** in \mathcal{L} is a set $H_e = H_e^+ \cap H_e^-$. A **polyhedron** P in \mathcal{L} is an intersection of closed halfspaces. The **rank** of a polyhedron P is the rank of the maximal covectors in P . An **accessible face** of P is a subset of P of the form $P \cap H_e$ for any $e \in E$ with $P \subseteq H_e^+$ or $P \subseteq H_e^-$. A **face** of P is any intersection of accessible faces, including the empty intersection, which is interpreted as P .

The definition of a **polyhedral complex** in \mathcal{L} and terminology for complexes is copied verbatim from Section 1, with \mathbb{R}^n replaced by \mathcal{L} . A set of covectors is **connected** if it induces a connected subgraph of the Hasse diagram of \mathcal{L} .

Theorem 5.1. *Fix $k \geq -1$ and let \mathcal{M} be a finite collection of polyhedra in an oriented matroid \mathcal{L} , each of rank greater than k . Suppose:*

- (i) For each $X \in \text{Supp}(\mathcal{M})$, the set $\{Y \in \text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M}) : Y \geq X\}$ is connected. (Here $\bigcap_k(\mathcal{M})$ is defined as in Section 1.)
- (ii) $M \cap N$ is a face of M and of N for all $M, N \in \mathcal{M}$ with $\text{rank}(M \cap N) > k$.

Then the collection of all polyhedra in \mathcal{M} and their faces is a polyhedral complex.

Proof. The oriented matroid version of Lemma 2.1 holds by a proof that is the same, except for a modification of the argument that $F \cap N$ is a face of $M \cap N$. For each accessible face F' of M containing F , we argue as in the proof of Lemma 2.1 that $F' \cap N$ is an accessible face of $M \cap N$. Since F is the intersection of all accessible faces of M containing it, we conclude that $F \cap N$ is an intersection of accessible faces of $M \cap N$, or in other words that $F \cap N$ is a face of $M \cap N$.

Let M and N be distinct polyhedra in \mathcal{M} , let F be the polyhedron $M \cap N$, and let d be the rank of F . Let X be a rank- d covector in F . Then the covectors in $\text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M})$ that are above X in the lattice \mathcal{L} form a connected set. Let Y be a covector of full rank in M and Z a covector of full rank in N . Then there is a path $Y = X_0, X_1, \dots, X_j = Z$ in $\text{Supp}(\mathcal{M}) \setminus \bigcap_k(\mathcal{M})$ consisting of covectors above X . Let $M_0 = M$ and $M_j = N$ and for each $i = 1, \dots, j-1$, let M_i be any polyhedron in \mathcal{M} with $X_i \in M_i$. The result is a sequence $M = M_0, \dots, M_j = N$ of polyhedra in \mathcal{M} , all containing X , such that, for each $i = 1, \dots, j$, the intersection $M_{i-1} \cap M_i$ is of rank greater than k . As in the proof of Theorem 1.2, we see that $F' = M_0 \cap \dots \cap M_j$ is a face of N . By construction, F' contains X and so has rank at least d . Since $F' \subseteq F$, it has rank exactly d .

Let $G = N \cap H$ be any accessible face of N containing F' . Then $F' \subseteq H$. Since F and F' are rank- d polyhedra with $F' \subseteq F$, we have $F \subseteq H$ and so $F \subseteq G$. Thus $F \subseteq F'$, so $F = F'$ is a face of N . By symmetry, F is a face of M . \square

If the rank of \mathcal{L} is n , then $\{X \in \mathcal{L} : \text{rank}(X) \in \{n-1, n\}\}$ is connected by [1, Proposition 4.2.3]. The following proposition is [1, Proposition 4.2.6].

Proposition 5.2. *Let P be an order ideal in \mathcal{L} generated by a set of covectors of full rank n . Then P is a polytope if and only if the following condition holds: If Y and Z are maximal in P and X is on a shortest path from X to Y in $\{X \in \mathcal{L} : \text{rank}(X) \in \{n-1, n\}\}$, then $X \in P$.*

In light of Proposition 5.2, the set $P \cap \{X \in \mathcal{L} : \text{rank}(X) \in \{n-1, n\}\}$ is connected, so we have the following corollary to Theorem 5.1.

Corollary 5.3. *Let \mathcal{M} be a finite collection of rank- d polyhedra in \mathcal{L} . Suppose:*

- (i) $\text{Supp}(\mathcal{M})$ is a polyhedron in \mathcal{L} ; and
- (ii) $M \cap N$ is a face of M and of N for all $M, N \in \mathcal{M}$ with $\text{rank}(M \cap N) \geq d-1$.

Then the collection of all polyhedra in \mathcal{M} and their faces is a polyhedral complex.

Paths, polygons, and path convexity in \mathcal{L} are defined as in the realizable case. Replacing Theorem 3.4 with [2, Theorem 2.4] and establishing, by the same proof, the analog of Lemma 3.3, we obtain the following theorem. (Cf. [1, Proposition 4.4.6].)

Theorem 5.4. *Every oriented matroid is path convex.*

Let F be a corank-2 face of a polyhedral complex \mathcal{C} in \mathcal{L} , not contained in the boundary of $\text{Supp}(\mathcal{C})$. (The **boundary** of a closed set U of covectors, i.e. an order ideal in \mathcal{L} , is the set of covectors X in U such that the principal order filter of \mathcal{L} generated by X is not contained in U .) Define $\text{Perp}(F)$ to be the rank-2

oriented matroid obtained by the deletion from \mathcal{L} of all $e \in E$ such that H_e does not contain F . Since every rank-2 oriented matroid is realizable, we can think of $\text{Perp}(F)$ as a fan. For each $M \in \mathcal{M}$ with $F \subset M$, define a cone in $\text{Perp}(F)$ as the set of covectors in $\text{Perp}(F)$ which arise by deletion from covectors in M . The fan $\mathcal{C}|_F$ is the collection of all such cones. The polygon property on a set \mathcal{E} of edges in the adjacency graph \mathcal{G} is now defined exactly as in the realizable case.

Theorem 5.5. *Let \mathcal{C} be a polyhedral complex in an oriented matroid \mathcal{L} such that $\text{Supp}(\mathcal{C})$ is a polyhedron. Then a subset \mathcal{E} of the edges of the adjacency graph \mathcal{G} is the edge set of a complex coarsening \mathcal{C} if and only if \mathcal{E} has the polygon property.*

The proof of Theorem 5.5 runs along the same lines as the proof of Theorem 1.1. First, the oriented-matroid analog of Lemma 4.3 holds by essentially the same proof, but instead of considering a line segment between points in M , we appeal to Proposition 5.2. The analog of Lemma 4.4 also holds by the same proof, except that instead of using a line segment $\overline{xx'}$ to construct a sequence of corank-1 covectors contained in F , we appeal to Proposition 5.2, applied to the restriction of \mathcal{L} to the hyperplane containing F .

Let \mathcal{C} be a polyhedral complex in \mathcal{L} with convex support. The notation \overline{X} will refer to the polyhedron which is the principal order ideal below X in \mathcal{L} . Thus \overline{X} should be thought of as the closure of X , while X should be thought of as the relative interior of \overline{X} . Let $\mathcal{C}(\mathcal{L})$ be the polyhedral complex in \mathcal{L} consisting of all polyhedra \overline{X} such that X is a covector in \mathcal{L} . Let $\overline{\mathcal{C}}$ be the subcomplex of $\mathcal{C}(\mathcal{L})$ consisting of faces contained in $\text{Supp}(\mathcal{C})$. Then \mathcal{C} coarsens $\overline{\mathcal{C}}$. As in the realizable case, we begin with a set \mathcal{E} of edges in the adjacency graph of \mathcal{C} , with the polygon property, and construct a set $\overline{\mathcal{E}}$ of edges of the adjacency graph of $\overline{\mathcal{C}}$. We then argue, along the same lines, that $\overline{\mathcal{E}}$ has the polygon property. The oriented-matroid analogs of Lemmas 4.3 and 4.4, with Corollary 5.3, imply that, $\overline{\mathcal{E}}$ is the edge set of a complex \mathcal{C}' coarsening $\overline{\mathcal{C}}$. The argument that \mathcal{E} is the edge set of \mathcal{C}' as a coarsening of \mathcal{C} extends to the oriented matroid case, and this completes the proof of Theorem 5.5.

Echoing the proof of Theorem 4.5, with $\mathcal{C}(\mathcal{A})$ replaced by $\mathcal{C}(\mathcal{L})$, we obtain an oriented matroid version of Tietze's convexity theorem. (The *interior* of $\text{Supp}(\mathcal{M})$ is $\text{Supp}(\mathcal{M})$ minus the boundary of $\text{Supp}(\mathcal{M})$.)

Theorem 5.6. *Let \mathcal{M} be a finite set of polyhedra of full rank in \mathcal{L} . Suppose:*

- (i) *The interior of $\text{Supp}(\mathcal{M})$ is connected; and*
- (ii) *For all X in the boundary of $\text{Supp}(\mathcal{M})$, there exists a closed halfspace H^+ bounded by a hyperplane H such that $X \in H$ and $\bigcup_{x \in M \in \mathcal{M}} M \subseteq H^+$.*

Then $\text{Supp}(\mathcal{M})$ is a polyhedron in \mathcal{L} .

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